Chapter 4

Eigenvalues and Eigenvectors

In this chapter we will look at matrix eigenvalue problems for \(2 \times 2\) and \(3 \times 3\) matrices. These are crucial in many areas of physics, and is a useful starting point for more general treatments of eigenvalue problems. They have a strong geometrical interpretation from the linear transformations discussed earlier in the course.

4.1 Homogeneous and Inhomogeneous matrix problems

In the previous chapters, simultaneous equations of the general form

\[
AX = C
\]

have been solved. These are called inhomogeneous equations, as the column matrix \(C\) on the right hand side is not zero. Homogeneous problems are of the form

\[
AX = 0,
\]

and admit the solution \(X = 0\). However, if \(\det A = 0\), then this equation may permit a family of solutions. In the two dimensional case this corresponds to the lines of the simultaneous equations being on top of each other. It is then possible to have a range of non-trivial solutions to the homogeneous equation. As we will now show, solving matrix eigenvalue problems is a homogeneous problem where we search for the eigenvalues that result in non-trivial solutions.

4.2 Matrix Eigenvalue problems

If for the matrix \(A\), the column matrix \(x\) and the scalar \(\lambda\)

\[
Ax = \lambda x
\]

then \(\lambda\) is said to be an eigenvalue of \(A\) and \(x\) an eigenvector of \(A\). To solve such equations we consider the \(2 \times 2\) system

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} = \lambda
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]
This set of equations have a trivial solution of \( x = 0, y = 0 \), but we are interested in the non-trivial solution. We can rearrange the previous equation to give
\[
\begin{pmatrix}
  a_{11} - \lambda & a_{12} \\
  a_{21} & a_{22} - \lambda
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = 0.
\] (4.3)

The only way we can have non-trivial solutions are if the determinant of the system is zero, i.e.
\[
\begin{vmatrix}
  a_{11} - \lambda & a_{12} \\
  a_{21} & a_{22} - \lambda
\end{vmatrix} = 0.
\] (4.4)

In this case we have some linearly dependent rows or columns as discussed in the previous chapter. Expanding out this determinant produces a quadratic equation
\[
(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0
\] (4.5)
which we can solve for the eigenvalues. This equation is known as the secular/characteristic equation. Let us consider a simple example here
\[
A = \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}
\] (4.6)

In this case we need the following determinant to be zero
\[
\begin{vmatrix}
  -\lambda & 1 \\
  1 & -\lambda
\end{vmatrix} = 0
\] (4.7)
\[
\Rightarrow \lambda^2 - 1 = 0
\] (4.8)
\[
\Rightarrow \lambda = \pm 1
\] (4.9)

**Eigenvectors**

Now that we have calculated the eigenvalues denoted by \( \lambda_i \), we can calculate their associated eigenvectors \( \mathbf{v}_i \). We simply go back to the eigenvalue equation
\[
A\mathbf{v}_i = \lambda\mathbf{v}_i
\] (4.10)
\[
\Rightarrow (A - \lambda I)\mathbf{v}_i = 0.
\] (4.11)

Before we solve this equation it should be born in mind that:

- \( \det(A - \lambda I) = 0 \) so the equations are linearly dependent,
- if \( \mathbf{v}_i \) is a solution, then so is \( \alpha\mathbf{v}_i \), for any multiple \( \alpha \),
- we can only determine the ratios of the components of \( \mathbf{v}_i \),
- this freedom is removed by giving *normalised* eigenvectors \( \mathbf{v}_i \), such that \( \mathbf{v}_i \cdot \mathbf{v}_i = 1 \).

These points will become clearer if we return to the example we started with above. We have found that the two eigenvalues are \( \lambda = \pm 1 \). For \( \lambda = 1 \) we can rewrite the eigenvalue equation as
\[
\begin{pmatrix}
  -1 & 1 \\
  1 & -1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = 0.
\] (4.12)
So we have that \(-x + y = 0\), and \(x - y = 0\) from the first and second row respectively. These two equivalent, so all we can obtain is the ratio \(x/y\). Our eigenvector is thus

\[
\mathbf{v}_+ = \begin{pmatrix} x \\ -x \end{pmatrix}
\]

(4.13)

The degree of freedom \(x\) can be removed by normalising this vector (i.e. choosing \(x\) such that the vector has length 1). We find that \(\mathbf{v}_+ \cdot \mathbf{v}_+ = 2x^2 = 1\), so if we set \(x = \frac{1}{\sqrt{2}}\) then \(\mathbf{v}_+\) is normalised, i.e.

\[
\mathbf{v}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

(4.14)

A similar analysis for \(\lambda = -1\) results in

\[
\mathbf{v}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

(4.15)

Diagonalisation

The remarkable thing about the two eigenvectors that we’ve just calculated is that they can be used to convert our matrix \(A\) into a diagonal matrix. If we form a matrix whose columns are the eigenvectors we have just found i.e.

\[
L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]

(4.16)

then if we calculate the new matrix \(A'\)

\[
A' = L^T A L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(4.17)

Geometrical interpretation

We earlier met the idea of a linear map in 2 dimensions. We can investigate the effect of these maps by calculating the eigenvalues and vectors of the corresponding matrix \(A\). For example, if we find that one of the eigenvalues is 1 then all the points in the direction of the corresponding eigenvector are invariant as they obey

\[
Ax = x.
\]

(4.18)

If we have a real, positive eigenvalue then all the points along the corresponding eigenvector are stretched by the corresponding factor. Let us consider an example

\[
A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.
\]

(4.19)

You should check that the eigenvalues of this matrix are 3 and -1 with corresponding eigenvectors \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\) respectively (note these are not normalised). The geometric interpretation of this is that along the line \(y = x\) the matrix stretched out space by a factor of 3, where as the matrix reflects the points along \(y = -x\) in the line \(y = x\).
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Example in 3 dimensions

These methods can be used for $3 \times 3$ matrices as well. For example consider the matrix

$$A = \begin{pmatrix} 0 & -1 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix}. \quad (4.20)$$

The eigenvalues of this matrix obey the usual eigenvalue equation

$$Ax = \lambda x \quad (4.21)$$

$$\Rightarrow (A - \lambda I)x = 0. \quad (4.22)$$

If we again want non-trivial solutions to this problem (i.e. solutions other than $x = 0$) then we must have that

$$|A - \lambda I| = 0. \quad (4.23)$$

For our example this equation is

$$\begin{vmatrix} -\lambda & -1 & -2 \\ 2 & 3 - \lambda & 2 \\ 1 & 1 & 3 - \lambda \end{vmatrix} = 0. \quad (4.24)$$

We can expand the determinant here as explained in the previous chapters. The result is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad (4.25)$$

$$\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \quad (4.26)$$

$$\Rightarrow \lambda = 1, 2, 3 \quad (4.27)$$

The eigenvectors can be found by substituting the eigenvalues back into the matrix. For example, consider the $\lambda = 1$ eigenvalue

$$\begin{pmatrix} -1 & -1 & -2 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad (4.28)$$

$$\Rightarrow -x - y - 2z = 0 \quad (4.29)$$

$$2x + 2y + 2z = 0 \quad (4.30)$$

$$x + y + 2z = 0 \quad (4.31)$$

$$\Rightarrow x = 1/\sqrt{2}, y = -1/\sqrt{2}, z = 0 \quad (4.32)$$

where we have been careful to normalise the eigenvector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Similarly for the other eigenvalues we find that $\lambda = 2$ has $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\lambda = 3$ has $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

The $3 \times 3$ matrix $A$ can also be diagonalised by constructing the $L$ matrix of eigenvectors

$$L = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & -1/\sqrt{3} \\ 0 & -1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix}. \quad (4.33)$$
If we then construct
\[ L^T AL = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \] (4.34)
i.e. we have *diagonalised* the matrix \( A \).

### 4.3 Higher dimensional eigenvalue problems

The eigenvalue problem of an arbitrarily sized square matrix is solved in the same way
\[ (A - \lambda I)x = 0 \] (4.35)
\[ \Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} - \lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0 \] (4.36)

This is in general an \( N \)th degree polynomial in \( \lambda \) with \( N \) roots for the eigenvalues. The set of eigenvalues is called the spectrum of the matrix. Each eigenvalue has a corresponding eigenvector \( v_i \) obtained from solving
\[ \begin{pmatrix} a_{11} - \lambda_i & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} - \lambda_i & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} - \lambda_i & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \end{pmatrix} = 0 \] (4.37)

As before there is no unique solution for the eigenvector, so we must solve for the ratios of the components and then use normalisation \( v \cdot v = 1 \) to have a unique solution. The reason for this arbitrariness is that if \( Av = \lambda v \) then \( A(cv) = \lambda(cv) \).

### Diagonalisation of a real symmetric matrix

Consider a real symmetric matrix \( A \), that is \( A = A^T \) and \( A = A^* \). Suppose we have calculated the eigenvalues \( \lambda_i \) and the associated *normalised* eigenvectors \( v_i \). Now it can be shown that each eigenvalue is real and each eigenvector is orthogonal (i.e. \( v_1 \cdot v_2 = 0 \) etc.). We will assume these results in what follows. Construct the matrix \( L \) using the eigenvectors as columns
\[ L = \begin{pmatrix} \uparrow & \uparrow & \vdots & \uparrow \\ v_1 & v_2 & \vdots & v_N \\ \downarrow & \downarrow & \vdots & \downarrow \end{pmatrix} \] (4.38)

If we now calculate \( L^T L \) then we find that because each eigenvector is orthogonal,
\[ L^T L = \begin{pmatrix} v_1 & \Rightarrow \\ v_2 & \Rightarrow \\ \vdots & \vdots & \vdots & \vdots \\ v_N & \Rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & \uparrow & \vdots & \uparrow \\ v_1 & v_2 & \vdots & v_N \\ \downarrow & \downarrow & \vdots & \downarrow \end{pmatrix} = I \] (4.39)
where $I$ is the identity matrix with 1s down the diagonal and zeroes elsewhere. We have shown that $L$ is an orthogonal matrix. If we multiply $A$ by $L$ then

$$AL = (A\mathbf{v}_1 A\mathbf{v}_2 \ldots A\mathbf{v}_N) = (\lambda_1 \mathbf{v}_1 \lambda_2 \mathbf{v}_2 \ldots \lambda_N \mathbf{v}_N)$$

(4.40)

$$= \begin{pmatrix} \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_N \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \ldots \\ 0 & \lambda_2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

(4.41)

So if we multiply this through by $L^T$ then we obtain

$$L^TAL = \begin{pmatrix} \lambda_1 & 0 & \ldots \\ 0 & \lambda_2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

(4.42)

which is a diagonal matrix with the elements equal to the eigenvalues of $A$.

What does this transformation correspond to? If we had an equation

$$Ax = y$$

(4.43)

$$AL^T \mathbf{x} = y$$

(4.44)

$$\Rightarrow \frac{L^T AL^T \mathbf{x}}{\mathbf{X}} = \frac{L^T y}{\mathbf{Y}}$$

(4.45)

So we have transformed to a new coordinate system with $\mathbf{X} = L^T \mathbf{x}$ and $\mathbf{Y} = L^T y$. The new coordinate axes are in the direction of the eigenvectors.

Note that the transformation

$$\Lambda = L^T AL$$

(4.46)

is known as a similarity transformation. The two matrices $A$ and $\Lambda$ are the same matrix in transformed coordinate systems.

### 4.4 Examples

**Normal modes**

An important application of eigenvalue problems in physics is the calculation of normal modes. Consider the mass spring system shown in the diagram below.
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$x_1$ and $x_2$ show the displacements from the equilibrium position of each of the masses. We can write down the equations of motion for the two masses as follows

\begin{align*}
    m\ddot{x}_1 &= -k_1 x_1 - k_2 (x_1 - x_2) \\
    m\ddot{x}_2 &= -k_1 x_2 - k_2 (x_2 - x_1)
\end{align*}

As we have seen we can rewrite this as a matrix equation

\begin{align*}
    \begin{pmatrix}
        \ddot{x}_1 \\
        \ddot{x}_2
    \end{pmatrix} &= \frac{1}{m} \begin{pmatrix}
        k_1 + k_2 & -k_2 \\
        -k_2 & k_1 + k_2
    \end{pmatrix}
    \begin{pmatrix}
        x_1 \\
        x_2
    \end{pmatrix}.
\end{align*}

The normal modes of the system have a constant frequency response at $\omega$. In this case we require that

\begin{align*}
    \ddot{x}_1 &= -\omega^2 x_1 \\
    \ddot{x}_2 &= -\omega^2 x_2
\end{align*}

i.e. SHM for each of the degrees of freedom. If we substitute this into our matrix equation, then we obtain

\begin{align*}
    Gx &= \lambda x
\end{align*}

where $\lambda = \omega^2$. This is an eigenvalue equation. We can solve this in the usual way, with the results

\begin{align*}
    \lambda_1 &= \omega^2_1 = \frac{k_1 + 2k_2}{m} \\
    v_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
        1 \\
        -1
    \end{pmatrix} \\
    \lambda_2 &= \omega^2_2 = \frac{k_1}{m} \\
    v_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix}
        1 \\
        1
    \end{pmatrix}
\end{align*}

Physically these modes correspond to an out of phase mode where the two masses oscillate in opposite directions, and an in phase mode where the two masses oscillate in the same directions. An arbitrary motion can be made by a superposition of these two modes, so is not in general SHM.

**Quadric forms**

Another application of eigenvalues and eigenvectors is in quadric forms. For example consider the equation

\begin{align*}
    2x^2 + 2y^2 + 2xy = 1.
\end{align*}

We can rewrite this equation using a matrix as follows

\begin{align*}
    \begin{pmatrix}
        x \\
        y
    \end{pmatrix} \begin{pmatrix}
        2 & 1 \\
        1 & 2
    \end{pmatrix} \begin{pmatrix}
        x \\
        y
    \end{pmatrix} = 1
\end{align*}

you should check this by multiplying out the matrices. Suppose now we define a new coordinate system such that

\begin{align*}
    \begin{pmatrix}
        X \\
        Y
    \end{pmatrix} = L^T \begin{pmatrix}
        x \\
        y
    \end{pmatrix}
\end{align*}
We could then write
\[
\begin{pmatrix} X & Y \end{pmatrix} L^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} L \begin{pmatrix} X \\ Y \end{pmatrix} = 1 \tag{4.58}
\]
where \(\Lambda\) is a diagonal matrix, with the eigenvalues as the entries along the diagonal. We simply have to calculate the eigenvalues and eigenvectors of
\[
\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} . \tag{4.59}
\]
These are given by
\[
\lambda_1 = 3 \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.60}
\]
\[
\lambda_2 = 1 \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} . \tag{4.61}
\]
So \(L\) is given by
\[
L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} . \tag{4.62}
\]
and our equation in its new coordinates is
\[
\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 1 \tag{4.63}
\]
\[
\Rightarrow 3X^2 + Y^2 = 1 \tag{4.64}
\]
This equation is much easier to sketch and is an ellipse. We can then deduce from our coordinate transformation that
\[
X = \frac{1}{\sqrt{2}} (x - y) \tag{4.65}
\]
\[
Y = \frac{1}{\sqrt{2}} (x + y) \tag{4.66}
\]
define the new coordinates.

### 4.5 Problems

**1A.** If
\[
S = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\]
show that \(S\) is an orthogonal matrix. Hence show that if
\[
P = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}
\]
then \(S P S^T\) is a diagonal matrix. Without further calculation give the eigenvalues of \(P\), and its corresponding eigenvectors.
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2B. Obtain the eigenvalues of the matrix
\[
\begin{pmatrix}
1 & 3 \\
2 & 2
\end{pmatrix}
\].

3B. Calculate the eigenvalues and eigenvectors of the following matrices
(a) \[
\begin{pmatrix}
0 & -1 \\
0 & -1
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
3 & 1 \\
2 & 1
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
-2 & 2 \\
-3 & 2
\end{pmatrix}
\]

4B. Obtain a cubic equation for the eigenvalues of the matrix
\[
A = \begin{pmatrix}
2 & 3 & 1 \\
3 & 1 & 2 \\
1 & 2 & 3
\end{pmatrix}
\]
and prove that the matrices
\[
B = \begin{pmatrix}
3 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix}
\]
and
\[
C = \begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{pmatrix}
\]
have the same eigenvalue equations as \(A\).

5B. Obtain eigenvalues and eigenvectors, normalised to unit length, for each of the following matrices
(a) \[
\begin{pmatrix}
1 & 4 & 5 \\
0 & 2 & 6 \\
0 & 0 & 3
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
2 & 3 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
2 & 2 & 0 \\
2 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
(d) \[
\begin{pmatrix}
1 & 1 + i \\
1 - i & 2
\end{pmatrix}
\]

6B. Find the eigenvalues and eigenvectors \((e_1, e_2, e_3)\) of the matrix
\[
A = \begin{pmatrix}
4 & -2 & 0 \\
-2 & 3 & -2 \\
0 & -2 & 2
\end{pmatrix}
\].
Let \(b = (6, -3, 0)\). By taking vector dot products with each of the eigenvectors in turn, find \(p_1, p_2\) and \(p_3\) such that \(b = p_1e_1 + p_2e_2 + p_3e_3\).
By using this decomposition, writing also \(x = q_1e_1 + q_2e_2 + q_3e_3\), solve the equation \(Ax = b\). Is the solution unique?

7B. Show that if \(A\) has eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\) then \(A^m\) (where \(m\) is a positive integer) has eigenvalues \(\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m\).

8C. Express the quadratic form \(x^2 + 16xy - 11y^2 = 1\) in matrix notation. By calculating the eigenvalues and eigenvectors of the matrix, write down the equation of the conic referred to the new axes along the directions of the eigenvectors.
Answers to Chapter 4 Problem

Note: you must attempt the problems before referring to the answers

1. If $S$ is orthogonal then $SS^T = I$.
   The diagonal elements should be the eigenvalues $4, -2$. The rows of $S$ can be read off to give
   the corresponding eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. $\lambda = 4, -1$

3. (Note the eigenvectors are not normalised here)

   (a) $-1, 0, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
   (b) $2 \pm \sqrt{3}, \begin{pmatrix} 1 + \sqrt{3} \\ 2 \\ 1 - \sqrt{3} \end{pmatrix}$
   (c) $\pm i\sqrt{2}, \begin{pmatrix} 2 - i\sqrt{2} \\ 3 \\ 2 + i\sqrt{2} \end{pmatrix}$

   Note you should take care when normalising these complex vectors as described in lectures

4. All matrices have the characteristic equation
   \[\lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0.\]

5. (Note the eigenvectors below are not normalised)

   (a) $3, 2, 1 \begin{pmatrix} 29 \\ 12 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
   (b) $2, 1, 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
   (c) $4, 1, 0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
   (d) $3, 0 \begin{pmatrix} 1 + i \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 - i \\ 1 \end{pmatrix}$

6. $6, 3, 0 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

   Note that $A$ is a singular matrix here (i.e. has zero determinant).

   $p_1 = 6, p_2 = -3, p_3 = 0$

   $q_1 = 1, q_2 = -1$, note that here $q_3$ is arbitrary as it has eigenvalue zero, so we can have any amount of $e_3$ in this problem.
7. We can diagonalise $A$ by using the matrix of eigenvectors, $L$

$$D = LAL^{-1}.$$  

If we rewrite this by multiplying it by $L$ from the right, and $L^{-1}$ from the left then

$$A = L^{-1}DL$$

Consequently $A^2 = L^{-1}DL^{-1}DL = L^{-1}D^2L$, so the diagonal matrix of eigenvalues now has diagonal elements $\lambda_1^2, \lambda_2^2$ and so on. Similarly for $A^m = L^{-1}DL^{-1}DL\ldots L^{-1}DL = L^{-1}D^mL$, so now the diagonal matrix has elements $\lambda_1^m, \lambda_2^m, \ldots$

8. 

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

Eigenvalues $-15, 5$ with eigenvectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

In new coordinate system the equation of this conic is

$$-15x^2 + 5y^2 = 1$$