Decay analysis with reservoir structures

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Contents:
• Introduction and some standard approaches
• Pseudomode method
• Connection via Fano diagonalization
• Where is the memory of a reservoir (quantum trajectories and pseudomodes)?
• Dynamic reservoir structures
Quantum system with environment

Quantum optical paradigms:

- Cavity system
- Photonic band-gap system
Fermi Golden rule

- Golden rule
  \[ \Gamma_{\alpha} = \frac{2\pi}{\hbar} |\hbar g_{\lambda}|^2 \frac{\rho_{\lambda}}{\hbar} \]

- Population decay
  \[ \Pi(t) = \Pi(0) e^{-\Gamma_{\alpha} t} \]
A first decay problem

Initial condition:

\[ |\Psi(0)\rangle = |1\rangle |0_\lambda \rangle \]

- TLS
- RWA

- Empty initial bath \( \Rightarrow \) Restricted Hilbert space
- Direct numerical simulation possible (nb. recurrences)

\[
\hat{H} = \sum_\lambda \hbar \omega_\lambda \hat{a}_\lambda^\dagger \hat{a}_\lambda + \frac{1}{2} \hbar \omega_1 (\hat{\sigma}^+ \hat{\sigma}^- - \hat{\sigma}^- \hat{\sigma}^+) \\
+ \sum_\lambda (\hbar g_\lambda^* \hat{a}_\lambda \hat{\sigma}^+ + \text{h.c.})
\]
Amplitude equations for dynamics

- State vector:

\[ |\Psi(t)\rangle = \tilde{c}_a(t)e^{-i\omega_1 t}|1\rangle|0\rangle + \sum_{\lambda} \tilde{c}_\lambda(t)e^{-i\omega_\lambda t}|0\rangle|1_\lambda\rangle \]

- Complex amplitude equations (\(\Delta_\lambda = \omega_\lambda - \omega_1\)):

\[
\begin{align*}
i \frac{d}{dt}\tilde{c}_a &= \sum_{\lambda} g_{\lambda}^* e^{-i\Delta_\lambda t}\tilde{c}_\lambda \\
i \frac{d}{dt}\tilde{c}_\lambda &= g_{\lambda} e^{i\Delta_\lambda t}\tilde{c}_a
\end{align*}
\]

- Integro-differential equation for atomic amplitude

\[
\frac{d}{dt}\tilde{c}_a(t) = -\int_0^t d\tau \tilde{G}(\tau)\tilde{c}_a(t-\tau)
\]

with memory kernel

\[
\tilde{G}(\tau) = \sum_{\lambda} |g_{\lambda}|^2 e^{-i\Delta_\lambda \tau} = \int d\omega_\lambda \rho(\omega_\lambda)|g_{\lambda}|^2 e^{-i\Delta_\lambda \tau}
\]

- **Reservoir structure function**: we let \(\rho_\lambda |g_{\lambda}|^2 = \frac{\Omega^2}{2\pi} D(\omega_\lambda)\)
Digression: Weisskopf-Wigner theory

- The bath is flat (or ‘fairly’ flat):

\[
\tilde{G}(\tau) = \sum_{\lambda} |g_{\lambda}|^2 e^{-i\Delta_{\lambda}\tau} = \int d\omega_{\lambda} \rho(\omega_{\lambda}) |g_{\lambda}|^2 e^{-i\Delta_{\lambda}\tau} \\
\rightarrow \rho |g|^2 \int d\omega_{\lambda} e^{-i\Delta_{\lambda}\tau} = \rho |g|^2 2\pi \delta(\tau)
\]

Integro-differential equation for atomic amplitude

\[
\frac{d}{dt} \tilde{c}_a(t) = -\int_0^t d\tau \tilde{G}(\tau) \tilde{c}_a(t - \tau) \\
= -2\pi \rho |g|^2 \int_0^t d\tau \delta(\tau) \tilde{c}_a(t - \tau) \\
= -\pi \rho |g|^2 \tilde{c}_a(t) \quad \text{n.b. factor } 1/2 \\
\equiv -\frac{\Gamma_a}{2} \tilde{c}_a(t)
\]

With population decay rate: \( \Gamma_a = 2\pi |g_{\lambda}|^2 \rho_{\lambda} \)
Excitation of bath modes

- Define excitation spectrum as $S(\omega_\lambda) = \rho_\lambda |c_\lambda|^2$
- Use ODE: $i \frac{d}{dt} \tilde{c}_\lambda = g_\lambda e^{i\Delta t} \tilde{c}_\alpha$
- For W-W:
  $$c_\lambda(t \to \infty) = \frac{g_\lambda^*}{(\omega_\lambda - \omega_0) + i\Gamma_\alpha/2}$$
  $$S(\omega, t \to \infty) = \frac{\rho(\omega)|g(\omega)|^2}{(\omega - \omega_0)^2 + (\Gamma_\alpha/2)^2}$$
  $$\sim \frac{\rho(\omega_0)|g(\omega_0)|^3}{(\omega - \omega_0)^2 + (\Gamma_\alpha/2)^2}$$
  $$\sim \frac{\Gamma_\alpha/2}{\pi [(\omega - \omega_0)^2 + (\Gamma_\alpha/2)^2]}.$$
Digression: Master Equations

\[
\frac{\partial \hat{\rho}(t)}{\partial t} = -i \left[ \hat{H}_I(t), \hat{\rho}(t) \right]
\]

Schrödinger Eqn. for system+bath; \( \rho = |\Psi\rangle \langle \Psi| \)

\[
\hat{H}_I(t) = \sum_\lambda \left( g_\lambda \hat{\sigma}^+ \hat{a}_\lambda e^{-i(\omega_\lambda - \omega_0)t} + g_\lambda^* \hat{\sigma}^- \hat{a}_\lambda^\dagger e^{i(\omega_\lambda - \omega_0)t} \right)
\]

Integrate and iterate …

\[
\hat{\rho}(t) = \hat{\rho}(t_0) - i \int_{t_0}^{t} \left[ \hat{H}_I(t'), \hat{\rho}(t') \right] \, dt'
\]
\[
\frac{\partial \hat{\rho}(t)}{\partial t} = -i \left[ \hat{H}_I(t), \hat{\rho}(t_0) \right] - \int_{t_0}^{t} \left[ \hat{H}_I(t), \left[ \hat{H}_I(t'), \hat{\rho}(t') \right] \right] \, dt'
\]

Approximation 1:

\[
\hat{\rho}(t) \approx \rho_S(t) \otimes \rho_B
\]

Bath is “large”, unaltered by interaction
Trace over bath for system operator

\[ \frac{\partial \hat{\rho}_S(t)}{\partial t} \approx -i \tau_B \left\{ \left[ \hat{H}_I(t), \hat{\rho}_S(t_0) \otimes \hat{\rho}_B \right] \right\} - \tau_B \int_{t_0}^{t} \left[ \hat{H}_I(t'), \left[ \hat{H}_I(t'), \hat{\rho}_S(t') \otimes \hat{\rho}_B \right] \right] dt' \].

Insert interaction Hamiltonian (+ make RWA)

\[ \frac{\partial \hat{\rho}_S(t)}{\partial t} \approx -\int_0^t \sum_\lambda |g_\lambda|^2 \left\{ e^{-i(\omega_\lambda - \omega_0)(t-t')} \left[ \hat{\sigma}^+ \hat{\sigma}^- \hat{\rho}_S(t') - \hat{\sigma}^- \hat{\rho}_S(t') \hat{\sigma}^+ \right] 
+ e^{i(\omega_\lambda - \omega_0)(t-t')} \left[ - \hat{\sigma}^- \hat{\rho}_S(t') \hat{\sigma}^+ + \hat{\rho}_S(t') \hat{\sigma}^+ \hat{\sigma}^- \right] \right\} dt' \]

Approximation 2: For a “broad” reservoir structure \( \rho_S(t') \approx \rho_S(t) \)

Integral contributes around \( t'=0 \); short correlation time for bath

\[ \frac{\partial \hat{\rho}_S(t)}{\partial t} \approx \Gamma_a \left\{ \hat{\sigma}^- \hat{\rho}_S(t) \hat{\sigma}^+ - \frac{1}{2} \hat{\sigma}^+ \hat{\sigma}^- \hat{\rho}_S(t) - \frac{1}{2} \hat{\rho}_S(t) \hat{\sigma}^+ \hat{\sigma}^- \right\} \]

This master equation is a weak coupling limit of PM theory.

End digression
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Pseudomode development

• Coupled amplitude equations ($\Delta_{\lambda} = \omega_{\lambda} - \omega_1$):

\[
i \frac{d}{dt} \tilde{c}_{\alpha} = \sum_{\lambda} g_{\lambda}^* e^{-i\Delta_{\lambda} t} \tilde{c}_{\lambda}
\]

\[
i \frac{d}{dt} \tilde{c}_{\lambda} = g_{\lambda} e^{i\Delta_{\lambda} t} \tilde{c}_{\alpha}
\]

• Integro-differential equation for atomic amplitude

\[
\frac{d}{dt} \tilde{c}_{\alpha}(t) = -\int_0^t d\tau \tilde{G}(\tau) \tilde{c}_{\alpha}(t - \tau)
\]

with memory kernel

\[
\tilde{G}(\tau) = \sum_{\lambda} |g_{\lambda}|^2 e^{-i\Delta_{\lambda} \tau} = \int d\omega \rho(\omega_{\lambda}) |g_{\lambda}|^2 e^{-i\Delta_{\lambda} \tau}
\]

• Reservoir structure function $D(\omega)$ or $\rho |g|^2$: we let

\[
\rho_{\lambda} |g_{\lambda}|^2 = \frac{\Omega^2}{2\pi} D(\omega_{\lambda})
\]

Normalisation:

\[
\frac{1}{2\pi} \int D(\omega) d\omega = 1
\]

• System behaviour depends on the reservoir structure function $D(\omega_\lambda)$ with weight $\Omega^2 = \int d\omega_\lambda \rho(\omega_\lambda) |g_\lambda|^2$

• Pseudomode idea based on considering poles of $D(\omega_\lambda)$ in the lower half complex $\omega_\lambda$ plane. Poles at $z_1, z_2, z_3 \ldots$, residues $r_1, r_2, r_3 \ldots$

• Extend $\omega$ to $-\infty$

• Evaluate contour for any meromorphic function

• Kernel $\rightarrow$

$$\tilde{G}(\tau) = -i\Omega^2 \sum_l r_l e^{-i(z_l - \omega_1)\tau}$$

Could solve for atomic $c_1(t)$, but instead …
• New kernel for same integro-differential equation:
\[ \frac{d}{dt} \tilde{c}_a(t) = - \int_0^t d\tau \tilde{G}(\tau) \tilde{c}_a(t - \tau) \quad \tilde{G}(\tau) = -i\Omega^2 \sum_l r_l e^{-i(z_l - \omega_1)\tau} \]

• Introduce effective amplitudes \( b_l \) and then
\[
\begin{align*}
 i \frac{dc_a(t)}{dt} &= \omega_1 c_a(t) + \sum_l K_l b_l(t) \\
 i \frac{db_l(t)}{dt} &= z_l b_l(t) + K_l c_a(t)
\end{align*}
\]

Where \( K_l = \Omega \sqrt{-ir_l} \) are PM couplings

• Atom-pseudomode system satisfies simple equations
• Pseudomodes replace continuum structure
• An exact description (within RWA …)
• Can derive exact master equations: examples follow…
Single pole \((l=1)\)

\[
D(\omega) = \frac{\Gamma}{(\omega - \omega_1)^2 + (\Gamma/2)^2}
\]

\[
z_1 = \omega_1 - i\frac{\Gamma}{2}
\]

\[
i \frac{d}{dt} \tilde{c}_\alpha = \sum_{\lambda} g_{\lambda} e^{-i\Delta \lambda t} \tilde{c}_\lambda
\]

\[
i \frac{d}{dt} \tilde{c}_\lambda = g_{\lambda} e^{i\Delta \lambda t} \tilde{c}_\alpha
\]

\[
i \frac{d}{dt} \tilde{c}_a = \omega_0 c_a + \Omega_0 b_1
\]

\[
i \frac{d}{dt} b_1 = (\omega_1 - i\frac{\Gamma}{2}) b_1 + \Omega_0 c_a
\]

Pseudomode master equation:

\[
\frac{d}{dt} \hat{\rho} = -i [H_0, \hat{\rho}] - \frac{\Gamma}{2} (\hat{a}^\dagger \hat{a} \hat{\rho} - 2 \hat{a} \hat{\rho} \hat{a}^\dagger + \hat{\rho} \hat{a}^\dagger \hat{a})
\]

with

\[
H_0 = \omega_0 (\hat{\sigma}_z + 1)/2 + \omega_c \hat{a}^\dagger \hat{a} + \Omega_0 (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+)
\]

\(\hat{\rho}\) is an enlarged atomic density matrix
Damped Rabi Oscillations

Solve the single pole pseudomode equations to give (resonance):

\[
c_\alpha(t) \approx e^{-\Gamma t/4} \cos \left( \frac{\Omega}{2} t \right)
\]

\[
|c_\alpha(t)|^2 \approx \frac{1}{2} e^{-\Gamma t/2} (1 + \cos(\Omega t))
\]

where \( \Omega = \sqrt{\Omega_0^2 - (\Gamma/2)^2} \)

- Exact and approximate solutions possible
- Numerical solution of equations with full bath possible, too.
- We can extract the bath \( c_k \) …
Vacuum Rabi Splitting

- Single pole example
- Excitation oscillates backwards and forwards between atom and reservoir.
- Reservoir excitation is an idealised spectrum.
- Final splitting is the Rabi frequency $\Omega$
- Width of final peaks is $\Gamma/2 (=\gamma)$
- Each Rabi oscillation increases the number of peaks in reservoir spectrum by one.
- Finite numerical bath has a recurrence time $=1/(\text{level-spacing})$
What do we learn?

- We can solve this example easily, e.g. by Laplace transform of the Integro-differential Eqn. – do we need to talk about “pseudomodes”?
- Answer: Yes – for other problems - if we want to talk about master equations.
- No perturbation theory was needed
- Link to a master equation: is this general …?
A simple band gap model (double pole)

\[ D(\omega) = W_1 \frac{\Gamma_1}{(\omega - \omega_c)^3 + (\Gamma_1/2)^2} - W_2 \frac{\Gamma_2}{(\omega - \omega_c)^2 + (\Gamma_2/2)^2} \]

Master equation:

\[
\frac{d}{dt} \hat{\rho} = -i [H_0, \hat{\rho}] - \frac{\Gamma'_1}{2} (\hat{a}_1^\dagger \hat{a}_1 \hat{\rho} - 2\hat{a}_1 \hat{\rho} \hat{a}_1^\dagger + \hat{\rho} \hat{a}_1^\dagger \hat{a}_1) - \frac{\Gamma'_2}{2} (\hat{a}_2^\dagger \hat{a}_2 \hat{\rho} - 2\hat{a}_2 \hat{\rho} \hat{a}_2^\dagger + \hat{\rho} \hat{a}_2^\dagger \hat{a}_2)
\]

with

\[
H_0 = \omega_0 (\hat{\sigma}_x + 1)/2 + \omega_c \hat{a}_1^\dagger \hat{a}_1 + \omega_c \hat{a}_2^\dagger \hat{a}_2 + \sqrt{W_1 W_2} (\Gamma_1 - \Gamma_2) \left( \frac{\hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger}{2} \right) + \Omega_0 (\hat{a}_2^\dagger \hat{\sigma}_- + \hat{a}_2 \hat{\sigma}_+) + \hat{\sigma}_x
\]

\[
\Gamma'_1 = \Gamma_1 \Gamma_2 \left( \frac{W_1}{\Gamma_1} - \frac{W_2}{\Gamma_2} \right), \quad \Gamma'_2 = W_1 \Gamma_1 - W_2 \Gamma_2
\]
• Atom-pseudomode system satisfies Lindblad form master equations (effect of augmentation)
• Pseudomodes can be coupled
• Can be problems e.g. with branch cuts
• Multiple excitations ...
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Three approaches to system description

**True-mode picture**
- **ATOMS**: $\hat{\sigma}_{k}\pm$
- **MODES**: $\hat{A}(\omega), \hat{A}^\dagger(\omega)$
- System: Non-Markovian
- Reservoir: Structured

**Quasi-mode picture**
- **ATOMS**: $\hat{\sigma}_{k}\pm$
- **MODES** (CAVITY): $\hat{a}_i, \hat{a}_i^\dagger$
- **MODES** (EXTERNAL): $\hat{b}(\Delta), \hat{b}^\dagger(\Delta)$
- New System: Markovian
- Reservoir: Flat

**Pseudo-mode picture**
- **ATOMS**: $\hat{\sigma}_{k}\pm$
- **MODES**: $\hat{\beta}_i, \hat{\beta}_i^\dagger$
Fano treatment: fields

True Mode Annihilation Operator – connection to internal and external QMs

\[ \hat{A}(\omega) = \sum_i \alpha_i(\omega) \hat{a}_i + \int d\Delta \rho_c(\Delta) \beta(\omega, \Delta) \hat{b}(\Delta) \]

Quasi-mode annihilation operators in terms of true modes:

\[ \hat{a}_i = \int d\omega \rho(\omega) \alpha_i^*(\omega) \hat{A}(\omega) \]

\[ \hat{b}(\Delta) = \int d\omega \rho(\omega) \beta^*(\Delta, \omega) \hat{A}(\omega) \]

\[ [\hat{A}(\omega), \hat{A}^\dagger(\omega')] = \delta(\omega - \omega')/\rho(\omega) \]

\[ [\hat{A}(\omega), \hat{H}_F] = \hbar \omega \hat{A}(\omega) \]
The inverse problem

• Given a density of states $\rho$ can we construct a pseudomode master equation?
• Represent actual $\rho g^2$ by model with poles (pseudomodes)
• Errors at high frequency (off resonant) in the model become discrepancies at long time scales.
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Time-local MEs and the pseudomodes

\[ |\Psi(t)\rangle = \tilde{c}_1(t)e^{-i\omega_1 t}|1\rangle|\ldots0\rangle + \sum_{\lambda} \tilde{c}_\lambda(t)e^{-i\omega_{\lambda} t}|0\rangle|\ldots1_\lambda\rangle \]

Non-Markovian dynamics in time-local Lindblad form (Breuer Petruccione Theory of open quantum systems, 2001)

\[
\frac{d\rho_A}{dt} = \frac{S(t)}{2i}[\sigma_+\sigma_-,\rho_A] + \gamma(t)\left[\sigma_-\rho_A\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho_A\}\right]
\]

\[
S(t) = -2 \text{ Im}\left\{\frac{\dot{c}_1(t)}{c_1(t)}\right\}, \quad \gamma(t) = -2 \text{ Re}\left\{\frac{\dot{c}_1(t)}{c_1(t)}\right\}.
\]

Preserves trace, positivity
Quantum jump simulations

Deterministic evolution:

\[ H = H_0 - \frac{i\hbar}{2} \sum_j \Gamma_j C_j^\dagger C_j. \]

\[
|\phi_\alpha(t + \delta t)\rangle = \left(1 - \frac{iH\delta t}{\hbar}\right)|\psi_\alpha(t)\rangle.
\]

\[
|\psi_\alpha(t)\rangle \rightarrow |\psi_\alpha(t + \delta t)\rangle = \frac{|\phi_\alpha(t + \delta t)\rangle}{|||\phi_\alpha(t + \delta t)\rangle||}.
\]

Jump process:

\[
|\psi_\alpha(t)\rangle \rightarrow |\psi_\alpha(t + \delta t)\rangle = \frac{C_j|\psi_\alpha(t)\rangle}{||C_j|\psi_\alpha(t)\rangle||}.
\]

Jump probability:

\[
p^j_\alpha(t) = \Gamma_j \delta t \langle \psi_\alpha(t)|C_j^\dagger C_j|\psi_\alpha(t)\rangle.
\]

Complete time step:

\[
\sigma_\alpha(t + \delta t) = (1 - p_\alpha)\frac{|\phi_\alpha(t + \delta t)\rangle\langle\phi_\alpha(t + \delta t)|}{1 - p_\alpha} + \sum_j p^j_\alpha \frac{C_j|\psi_\alpha(t)\rangle\langle\psi_\alpha(t)|C_j^\dagger}{\langle\psi_\alpha(t)|C_j^\dagger C_j|\psi_\alpha(t)\rangle}.
\]
Non-Markovian quantum jumps (Piilo et al 2008)

Why needed?

\[
\frac{d\rho_A}{dt} = \frac{S(t)}{2i}[\sigma_+\sigma_-\rho_A] + \gamma(t) \left[ \sigma_-\rho_A\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-\rho_A\} \right]
\]

\(\gamma(t)\) can be negative!

Simulation problems: negative probabilities
… use new NMQJ approach
Time-local MEs and the pseudomodes

Breuer Petruccione master equation

\[ \frac{d\rho_A}{dt} = \frac{S(t)}{2i} [\sigma_+ \rho A \sigma_- - \frac{1}{2} \{\sigma_+, \rho A\} + \gamma(t) \left[ \rho A \sigma_+ - \frac{1}{2} \{\sigma_+, \rho A\} \right] \]

\[ S(t) = -2 \text{Im} \left\{ \frac{c^*_1(t)}{c_1(t)} \right\}, \quad \gamma(t) = -2 \text{Re} \left\{ \frac{c^*_1(t)}{c_1(t)} \right\}. \]

Pseudomode master equation (Lorentzian structure)

\[ \frac{d\rho}{dt} = -i[H_0, \rho] - \frac{\Gamma}{2} [a^+ \rho a - 2\rho a^+ a + \rho a^+ a], \]

\[ H_0 = \omega_0 \sigma_+ \sigma_- + \omega_c a^+ a + \Omega_0 [a^+ \sigma_- + a \sigma_+]. \]

Density matrix \( \rho \) is in an enlarged space. Trace out the PM:

\[ \frac{d\rho_A}{dt} = \frac{A(t)}{2i} [\sigma_+ \rho_A \sigma_- - \frac{1}{2} \{\sigma_+, \rho_A\} + B(t) \left[ \sigma_+ \rho_A \sigma_- - \frac{1}{2} \{\sigma_+, \rho_A\} \right], \]

The connection

\[ A(t) = S(t) \]

\[ B(t) = \gamma(t) \]

Link to pseudomodes yields interpretation with

\[ \frac{d|b_1(t)|^2}{dt} + \Gamma |b_1(t)|^2 = \gamma(t) |c_1(t)|^2 \]

Normal QJ simulation possible.

Environment division:

- Memory part
- Non-memory part
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Time-dependent reservoir structures

- Cavity realisation: A time dependent position of the mirror affects mode structure

A: Essentially straightforward and previously studied (GSA+…).

\[ \hat{H}_I(t) = \sum_k g_k(t) \left[ \hat{\sigma} - \hat{b}_k^\dagger \exp \left( -i \int_0^t \left[ \omega_0 - \omega_k(\tau) \right] d\tau \right) + \hat{\sigma}^+ \hat{b}_k \exp \left( +i \int_0^t \left[ \omega_0 - \omega_k(\tau) \right] d\tau \right) \right] \]

B: Could lead to a different kind of dynamic structure. Quantify and explore.
Atomic dynamics

Solve the Schrödinger equation to give:

\[
\begin{align*}
    \frac{i}{\partial t} \frac{\partial c_{\alpha}(t)}{\partial t} &= \sum_{k} g_{k}(t) \exp \left( +i \int_{0}^{t} [\omega_{0} - \omega_{k}(\tau)] d\tau \right) c_{k}(t) \\
    \frac{i}{\partial t} \frac{\partial c_{k}(t)}{\partial t} &= g_{k}(t) \exp \left( -i \int_{0}^{t} [\omega_{0} - \omega_{k}(\tau)] d\tau \right) c_{a}(t)
\end{align*}
\]

Eliminate Bath Modes

\[
\frac{\partial c_{\alpha}(t)}{\partial t} = - \int_{0}^{t} K(t, t') c_{\alpha}(t') \, dt'
\]

Memory Kernel

\[
K(t, t') = \sum_{k} g_{k}(t) g_{k}(t') \exp \left( +i \int_{t'}^{t} [\omega_{0} - \omega_{k}(\tau)] d\tau \right)
\]
A chirped bath model

Macroscopic structure remains static while a ‘wind’ of modes pass through the resonance ⇒ linear chirp of bath modes: \( \omega(k,t) = \omega_k + \chi t \)

\[
\rho_k g_k(t) g_k(t') = \frac{(\Omega_0/2)^2 \gamma}{\pi \sqrt[\gamma^2 + (\omega_0 - \omega_k(t))^2] [\gamma^2 + (\omega_0 - \omega_k(t'))^2]}
\]

- A new feature of this model is the presence of TWO-TIME reservoir structure function
- Branch cuts ⇒ mostly numerical approach …
Different regimes of interest

\[ \xi = \frac{\pi \alpha \Omega_0}{4D^2} \]
(Scaled chirp-rate)

Linington & Garraway, J Phys B (2006), \( D \to \Omega_0/2 \)
Population Recycling

...if the mode-frequencies are modulated at this characteristic rate, then the left-hand Rabi peak may be repeatedly brought back onto resonance.

Rabi-oscillations become stable due to recycling of reservoir population.
High chirp-rate – Markov limit

(Analytic approach)

In the high-chirp limit ($\xi >> 1$), each bath mode is effectively coupled to the atom for only a very short time; there is no time for memory effects.

\[
\frac{\partial c_a(t)}{\partial t} \approx -c_a(t) \frac{\Omega_0^2 \gamma}{4\pi} \int_0^t dt' \int_{-\infty}^{\infty} d\omega_k \frac{\exp \left[ +i (\omega_0 - \omega_k) t - \frac{i\chi t^2}{2} \right]}{\sqrt{\left( \gamma^2 + (\omega_0 - \omega_k - \chi t)^2 \right)}}
\]

\[
\times \frac{\exp \left[ -i (\omega_0 - \omega_k) t' + \frac{i\chi t'^2}{2} \right]}{\sqrt{\left( \gamma^2 + (\omega_0 - \omega_k - \chi t')^2 \right)}}
\]

\[
\frac{\partial c_a(t)}{\partial t} \approx -c_a(t) \Gamma(t) \\
\text{Approximate Markovian form – even for strong coupling}
\]

\[
\Gamma_\infty = \lim_{t \to \infty} \{ \Gamma(t) \} = \frac{\Omega_0^2 \gamma}{8\pi \alpha \omega_0} \left| K_0 \left( \frac{i\gamma^2}{4\alpha \omega_0} \right) \right|^2
\]

Can extract energy from the cavity on a very short time-scale!
Case 2: Oscillating frequency manipulation

\[ \omega_k(t) = \omega_k + d \sin(\Omega t) \]

Non-linear phase-term:

\[ \exp \left( +i[\omega_0 - \omega_k]t + i\frac{d}{\Omega} [1 - \cos(\Omega t)] \right) \]

Can be Fourier-decomposed as a sum of simple phase-factors:

\[ \sum_{l=-\infty}^{\infty} (-i)^l J_l(d/\Omega) \exp \left( i[\omega_k - \omega_0 - l\Omega]t + i\frac{d}{\Omega} \right) \]
Two equivalent ways of thinking about the problem...

...either the atom is coupled to a single bath with time-dependent couplings and static frequencies, or the atom is coupled to a collection of baths with time-dependent frequencies.
Observable effects

Weak coupling: Inhibition of decay-rate by a factor of up to 1000

Strong coupling: Enhancement of decay-rate by up to a factor 15, (or modest suppression).

Modulation frequency:
\[ \Omega/2\pi \leq 10^7 \text{s}^{-1} \]

Linewidth:
\[ \gamma/2\pi \leq 10^6 \text{s}^{-1} \]

Coupling strength:
\[ D/2\pi \leq 4 \times 10^7 \text{s}^{-1} \]

(Numbers borrowed from PRL 93, 233603 (2004) – Kimble group)
Summary

- **Pseudomodes:**
  - Pseudomode: the part of the reservoir structure which retains a memory.
  - Meromorphic reservoir structures lead to standard master equations
  - Multiple excitations covered and checked
  - No representation when branch cuts present – does it matter?
  - Expect application to local density of states (reservoir structure)

- **Time dependent reservoir structures:**
  - At present pseudomodes only useful in perturbative limit
  - High chirp → Markovian decay (which can be fast!)
  - Enhanced and inhibited decay via $\chi$ seen for linear and oscillatory chirp
  - Rich behaviour …

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References


