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Shell Model for Open Quantum Systems

- Formulation of the problem
- Shell Model for weakly bound and unbound states
- Salient continuum-coupling phenomena – few examples
Atom: electron shells

Nucleus: proton/neutron shells

Shell Model of Atoms

Shell Model of Nuclei
No shell closure for N=8 and 20 for drip-line nuclei; new shells at 14, 16, 32… (‘monopole migration’)

Old paradigms, universal ideas, are not correct: near the drip lines, nuclear structure may be dramatically different.
If continuum space is not considered, the system becomes closed (Shell Model,...)

Network of coupled many-body systems
When can we talk about “existence” of an unbound system?

\[ T_{1/2} = \ln 2 \frac{\hbar}{\Gamma} , \quad \hbar = 6.58 \cdot 10^{-22} \text{ MeV} \cdot \text{sec} \]

\[ T_{s.p.} \approx 3 \cdot 10^{-22} \text{ sec} = 3\text{baby sec} . \]

\[ T_{1/2} \gg T_{s.p.} \]

\[ \Gamma \ll 1\text{MeV} \]
Threshold effects: at the intersection between mean-field and collective energy scales

Paradigm of Nuclear Shell Model is incorrect
The nucleus is a correlated open quantum many-body system
Environment: the continuum of decay channels
Instability of SM eigenstates at channel threshold

Pairing correction to single-particle eigenstates

Continuum coupling correction to shell model eigenstates

Admixture of many-body continuum states with $E > E_{th}$

Instability of a single-particle motion in a potential with a pair deformation

Admixture of single-particle configurations with $e > \lambda$
Shell Model in the Complex Energy Plane (Gamow Shell Model) (Rigged Hilbert space formulation)

Quasi-stationary open quantum system extension of the SM for well-bound systems

\[ i\hbar \frac{\partial}{\partial t} \Phi(r,t) = \hat{H} \Phi(r,t) \quad \Phi(r,t) = \tau(t) \Psi(r) \]

\[ \hat{H} \Psi = \left( e - i \frac{\Gamma}{2} \right) \Psi \quad \tau(t) = \exp \left( -i \left( e - i \frac{\Gamma}{2} \right) t \right) \]

\[ \Psi(0,k) = 0 , \quad \begin{cases} 
\Psi(\tilde{r},k) \rightarrow O_l(kr) \quad r \rightarrow \infty \\
\Psi(\tilde{r},k) \rightarrow I_l(kr) + O_l(kr) \quad r \rightarrow \infty
\end{cases} \]

\[ k_n = \sqrt{\frac{2m}{\hbar^2} \left( e_n - i \frac{\Gamma_n}{2} \right)} \quad \text{(poles of the S-matrix)} \]

- **Bound states**: \( k_n = i\kappa_n \)
- **Antibound states**: \( k_n = -i\kappa_n \)
- **Resonances**: \( k_n = \pm \gamma_n - i\kappa_n \)

Only bound states are integrable!
Resonances are genuine intrinsic properties of quantum systems but they do not belong to the Hilbert space

Completeness relation for one-body states:

$$\sum_n |u_n \langle \tilde{u}_n| + \int_{L^+} |u_k \langle \tilde{u}_k| dk = 1$$

Completeness relation for many-body states:

$$\sum_k |SD_k \langle SD_k| \approx 1$$

Euclidean inner product

$$\langle u_n | u_n \rangle = \int_0^\infty dr u_n^*(r) u_n(r) \quad \overset{r \to \infty}{\longrightarrow} e^{2k_2 r}$$

RHS inner product

$$\langle \tilde{u}_n | u_n \rangle = \int_0^\infty dr \tilde{u}_n^*(r) u_n(r) \quad \overset{r \to \infty}{\longrightarrow} e^{2i(k_1 - ik_2)}$$
Gamow Shell Model

\[ H = \sum_{i=1}^{A} \left( \frac{p_i^2}{2m} + U_i \right) + \sum_{i<j} V_{ij} + \frac{p_ip_j}{(A_c + 1)m} \]

(i) two-step diagonalization: \[ \{\Psi_0^J\} \rightarrow \{\Psi_J\} \]
- selection of states: \[ \max \left| \langle \Psi_0^J | \Psi_i^J \rangle \right| \]

(ii) Density Matrix Renormalization Group method

J. Rotureau et al., PRL 97, 110603

SM respecting unitarity in weakly-bound/unbound states is built on a skeleton of the S-matrix and the many-body completeness relation

N. Michel, W. Nazarewicz, M. Ploszajczak, T. Vertse,
In mathematics, a **rigged Hilbert space** (Gel’fand triple, nested Hilbert space, or equipped Hilbert space) is a construction designed to link the distribution and square-integrable aspects of functional analysis. Such spaces were introduced to study spectral theory in the broad sense. They can bring together the ‘bound state’ (eigenvector’ and ‘continuous spectrum’, in one place. The also provide a natural formulation of the Dirac ket-formalism and time-asymmetric aspects of quantum world.

Mathematical foundations in the 1960s by Gel’fand et al. who combined Hilbert space with the theory of distributions. Hence the rigged Hilbert space, rather than the Hilbert space alone, is the natural mathematical setting of Quantum Mechanics.

I.M. Gel’fand and N.J. Vilenkin, Generalized Functions, vol. 4: Some Applications of Harmonic Analysis. Rigged Hilbert Spaces  
Academic Press, New York, 1964
Salient continuum-coupling phenomena

- Anti-odd-even staggering in odd-Z (N) isotopic chains
- Threshold effects in configuration mixing (spectroscopic factors, mirror symmetry breaking, asymptotic normalization coefficients of 1N-overlap integrals,...)
- Alignment of near-threshold states with the decay channel
- Segregation of time-scales in the continuum
- Phase rigidity variations
- Level degeneracies
Anti-odd-even staggering of $E_{\text{corr}}$

Continuum-coupling correction to binding energies

Fluorine (Z=9) isotopes

$E_n^{(th)} = 0$

$E_n^{(th)} = 4\,\text{MeV}$

$E_n^{(th)\exp}$

J. Okolowicz, et al., Physics Reports 374, 271
Strong, near-threshold variations of the continuum-coupling energy correction for certain states (‘Hermitian’ configuration mixing)
Configuration mixing in weakly bound/unbound many-body states

\[ S^2 = \int u^2_{ij}(r) dr = \sum_{B} \langle \Psi_A^J | a_{ij}^+(B) | \Psi_A^{J-1} \rangle \]

\[ \langle \hat{\Psi}_g.s. | [ ^5\text{He}(\text{g.s.}) \otimes p_{3/2}^\pm ]^0 \rangle \]

\[ \langle ^6\text{He}(\text{g.s.}) | [ ^5\text{He}(\text{g.s.}) \otimes p_{3/2}^\pm ]^0 \rangle \]

Analogy with Wigner threshold phenomenon for reaction cross-sections

E.P. Wigner, Phys. Rev. 73, 1002


\[ Y(b,a)X : \sigma_\ell \sim k^{2\ell-1} \]

\[ X(a,b)Y : \sigma_\ell \sim k^{2\ell+1} \]

\[ (-S_n)^{\ell-1/2} \quad \text{for} \quad S_n < 0 \]

\[ (-S_n)^{\ell+1/2} \quad \text{for} \quad S_n > 0 \]

bound-state structure dominates

non-perturbative behavior

bound

weakly bound/unbound

independent of the s.p. basis
Configuration mixing in mirror systems

\[ ^{2}\text{He}_{3} + n \rightarrow 1867 \quad 2^+ \]
\[ ^{2}\text{He}_{2} + 2n \rightarrow 1797 \quad 2^+ \]
\[ ^{3}\text{He}_{2} + p \rightarrow 973 \quad 0^+ \]
\[ ^{5}\text{Li}_{2} + p \rightarrow 595 \quad 0^+ \]
\[ ^{6}\text{He}_{4} \]
\[ ^{6}\text{Be}_{2} \]
\[ ^{2}\text{He}_{2} + 2p \rightarrow -1371 \quad 0^+ \]

For experimental \( S_{1n}/S_{1p}, S_{2n}/S_{2p} \):

\[ S_{^{6}\text{He}}^{2}\left[ 0^+ \right] = 0.88 - i0.386 \]
\[ S_{^{6}\text{Be}}^{2}\left[ 0^+ \right] = 1.057 - i0.181 \]
Alignment of near-threshold states with decay channels

$2^+ \rightarrow \pi(s_{1/2}) + 5/2^+_1 \oplus \pi(s_{1/2}) \oplus 5/2^+_1$

$2^+_2$ aligns with the decay channel $p^{+17}F(5/2^+_1) \rightarrow ^{18}\text{Ne}(2^+_2)$

Halo and cluster states close to the threshold

**α cluster states**

- $^8\text{Be} + \alpha$
  - $0^+$: 7654
  - $0^+$: 7366
- $\sim (0p)^8$
  - $2^+$: 4439

- $^{12}\text{C}$

**Halo states**

- $^{10}\text{Li} + n$
  - $3/2^-$: 325
  - $3/2^-$: 300
- $^{11}\text{Li}$
Phase rigidity and configuration mixing

$^{16}\text{Ne}$ \hspace{1cm} J^\pi = 0^+$

\[ r = |\rho^2| \]
\[ \rho = \frac{\int d\mathbf{r} \psi(\mathbf{r})^2}{\int d\mathbf{r} |\psi(\mathbf{r})|^2} = e^{2i\psi} \frac{\int d\mathbf{r} |\psi_\rho(\mathbf{r})|^2 - |\psi_\imath(\mathbf{r})|^2}{\int d\mathbf{r} |\psi_\rho(\mathbf{r})|^2 + |\psi_\imath(\mathbf{r})|^2} \]

Exceptional strings close to the threshold

$^{16}\text{Ne}$ \hspace{1cm} $J^\pi = 0^+$

Strong configurations mixing near threshold due to the proximity of exceptional strings
Segregation of time scales in the continuum

\[ J^\pi = 0^+, T = 0 \] states in \(^{24}\text{Mg}\), 10 channels

\begin{align*}
\Gamma[\text{a.u.}] & \quad E[\text{a.u.}] \\
\text{intermediate coupling} & \quad \text{strong coupling}
\end{align*}

closed cavity

‘Bound’ states in the continuum

opened cavity
Physics of open nuclei is demanding!

Interactions
- Poorly-known spin-isospin components come into play
- Long isotopic chains crucial

Many-body Correlations
- Nuclei are open quantum systems
- Exotic nuclei have low-energy decay thresholds
- Coupling to the continuum important
  - Virtual scattering
  - Unbound states
  - Impact on in-medium Interactions

Open Channels
- \( ^7\text{H}, ^{11}\text{Be}, ^{42}\text{Si}, ^{45}\text{Fe}, ^{101}\text{Sn}, ^{141}\text{Ho} \)

Configuration interaction
- Mean-field concept often questionable
- Asymmetry of proton and neutron Fermi surfaces gives rise to new couplings
- New collective modes; polarization effects
Thank you!
Part II
Quantum tunneling in the driven many-body system


Tunneling is a non-perturbative phenomenon of purely quantum origin

**Static SU(2) model**

\[ \varepsilon_1 = +0.5 \]

\[ \varepsilon_1 = -0.5 \]

\[ \hat{H} = \sum_{k=1}^{2} \varepsilon_k \left( \sum_{n=1}^{N} a_{nk}^+ a_{nk} \right) - \frac{1}{2} \sum_{k,l=1}^{2} V_{kl} \left( \sum_{n=1}^{N} a_{nk}^+ a_{nl} \right)^2 \]

\[ V_{kl} = V \left( 1 - \delta_{kl} \right) \quad V \geq 0 \]

\[ G_{kl} = \sum_{n=1}^{N} a_{nk}^+ a_{nl} \Rightarrow \begin{cases} K_0 = \frac{1}{2} (G_{22} - G_{11}) \\ K_+ = G_{21} \\ K_- = G_{12} \end{cases} \]

\[ \hat{H} = \varepsilon K_0 - \frac{1}{2} V \left( K_+^2 + K_-^2 \right) \quad \varepsilon = \varepsilon_2 - \varepsilon_1 \]
Classical limit

\[ |\psi_{SD}(z)\rangle \equiv |z\rangle = \exp(z^* G_{21})|0\rangle \Rightarrow \lim_{N\to \infty} \Psi_{g.s.}^{(SU(2))} = \Psi_{HF}^{(SU(2))} \]

\[
(z, z^*) \rightarrow \left( \beta = \frac{1}{\sqrt{1 + z^* z}}, \beta^* = \frac{1}{\sqrt{1 + z z^*}} \right) \rightarrow \left( q = \sqrt{\frac{1}{2}} (\beta + \beta^*), \quad p = \sqrt{\frac{1}{2}} (\beta - \beta^*) \right)
\]

\[
\mathcal{H}_{cl} = \frac{\langle \psi_{SD} | \hat{H} | \psi_{SD} \rangle}{N\varepsilon} = -\frac{1}{2} + \frac{1}{2} (1 - \kappa) q^2 + \frac{1}{2} (1 + \kappa) p^2 + \frac{1}{4} \kappa (q^4 - p^4) \quad \kappa = \frac{V(N-1)}{\varepsilon}
\]

\[\Delta V = \frac{\kappa^2 - 8 \kappa + 1}{4 \kappa} \sim \kappa\]

\[V(q) = \mathcal{H}_{cl}(q, p = 0) = \frac{1}{4} \kappa q^4 + \frac{1}{2} (1 - \kappa) q^2 - \frac{1}{2}\]

\[\pm q_{eq} = \pm \sqrt{(\kappa - 1) / \kappa}\]
The time-dependent SU(2) model

\[ V(t) = V_0 + \frac{\alpha \varepsilon}{N-1} \sin^9(\beta t) \Leftrightarrow \kappa(t) = \kappa_0 + \alpha \sin^9(\beta t) \]

Driving does not change the symmetry of the unperturbed (static) Hamiltonians: \( \hat{H}, \hat{H}_{cl} \)

Solution:

\[
\begin{cases}
\hat{H}(t) = H\left(t + \frac{2\pi n}{\beta}\right) \\
\hat{H}_{cl}(t) = H_{cl}\left(t + \frac{2\pi n}{\beta}\right)
\end{cases} 
\]

\( n = 0, \pm 1, \pm 2, \ldots \)

Floquet theorem:

\[ \Psi(t) = \sum_k c_k \Omega_k(t) = \sum_k c_k \exp(-ie_k t) \Phi_k(t) \]

\[ \Phi_k(t) = \Phi_k\left(t + \frac{2\pi n}{\beta}\right) \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ \Omega_k(t) \equiv \exp(-ie_k t) \Phi_k(t) = \exp(-ie_{k,l} t) \Phi_{k,l}(t) \quad e_{k,l} = e_k + l\beta \]
For each $k$ there exists an infinite number of quasi-energies $e_{k,l}$, however number of independent quasi-energy classes is always equal to the number of eigenenergies in the corresponding stationary case

\[
\begin{align*}
\Omega_k(t) &= \sum_n \Phi_{k,n} e^{i\beta t} \exp(i e_k t) \\
\hat{H}(t) &= \sum_n \hat{H}_{(n)} e^{i\beta t} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \\
\sum_l \left( \hat{H}_{(n-l)} + n\beta \delta_{l,n} \right) \Phi_{k,l} &= e_k \Phi_{k,n} \quad n, l = 0, \pm 1, \pm 2, \pm 3, \ldots
\end{align*}
\]
In the absence of time-dependent driving:

\[ |\Psi(t = 0)\rangle = |q_{eq}, p = 0\rangle \equiv \sqrt{\frac{1}{2}} (|\Psi_0^{(+)}\rangle + |\Psi_0^{(-)}\rangle) \Rightarrow |\Psi(t)\rangle \equiv \sqrt{\frac{1}{2}} \left( \exp\left(-iE_0^{(+)t}\right)|\Psi_0^{(+)}\rangle + \exp\left(-iE_0^{(-)t}\right)|\Psi_0^{(-)}\rangle \right) \]

the wave packet oscillates coherently with the frequency: 

\[ T_{osc} = \frac{2\pi}{|E_0^{(+)} - E_0^{(-)}|} \]

Evolution of the wave packet in the presence of time-periodic driving depends on the spectrum of quasi-energies

\[ W_\Psi(q,p;t) = \left| \langle \Psi_{SD}(q,p)|\Psi(t)\rangle \right|^2 = \left| \langle q,p|\Psi(t)\rangle \right|^2 \]

\[ \kappa(t) = \kappa_0 + \alpha \sin^9(\beta t) \]
\[ \kappa_0 = 5 \]
\[ \alpha = 4.75 \]
\[ \beta = 9 \approx 1.3\omega_0 \quad \omega_0 = \sqrt{2(\kappa_0^2 - 1)} \]

Tunneling remains coherent – small number of important quasi-energy \( \{e_{k,l}\} \) eigenstates in \( \Psi_{SD}(q_{eq}, p = 0) \)
The quasi-energy spectrum depends on the external environment and can be tuned
Coherent suppression of the tunneling

Outlook:
- Tunneling in systems with GOE (GOU) and Poisson quasi-energy spectra statistics
- Fast, incoherent transitions in ‘chaotic’ time-periodic driven many-body systems

Could we control nuclear decay rates in a foreseeable future?